

Friday, November 14, Lecture 32: Convolution, Dirac Deltas

Last time... Convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

Convolution $\xrightarrow{F}$ Multiplication
 $\xleftarrow{F^{-1}}$

$$\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g} \quad \leftarrow f * g = F^{-1}(\sqrt{2\pi} \hat{f} \hat{g})$$

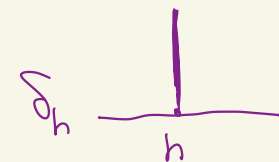
call f a test function.

Dirac Delta: If f is "nice enough" (usually, f is infinitely diff. and $f^{(n)}(x) \rightarrow 0$ as $x \rightarrow \infty$)

$$\int_{-\infty}^{\infty} \delta_h(x) f(x) = f(h)$$

$$(\delta_h * f)(x) = f(x-h)$$

$\uparrow$ if $h=0$,



$$\delta * f = f.$$

First: a concrete application of convolution.

Back to the heat eqn, say on an infinite rod, so $x \in \mathbb{R}$ position
 $t \in \mathbb{R}$ time

$$(*) \quad \frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T$$

$$T(x, 0) = f(x) \quad \text{Initial condition}$$

At each fixed time t , T is a function $\mathbb{R} \rightarrow \mathbb{R}$.
 So we can take a Fourier transform

$$\hat{T}(k, t) = \int_{-\infty}^{\infty} T(x, t) e^{-ikx} dx.$$

plug in, noting

$$\begin{aligned} \widehat{\frac{\partial T}{\partial t}} &= \widehat{\frac{\partial^2 T}{\partial x^2}} \quad \text{use } (*) \\ \widehat{\left(-i \frac{\partial T}{\partial x}\right)} &= k \hat{T} \quad \text{write} \quad \rightarrow = (ik)^2 \hat{T} \\ &= -k^2 \hat{T} \end{aligned}$$

This is an easy ODE to solve:

$$\begin{aligned} \hat{T}(k, t) &= e^{-k^2 t} \hat{T}(k, 0) \\ &= e^{-k^2 t} \hat{f}(k). \end{aligned}$$

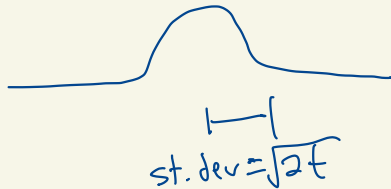
So, to recover $T(x, t)$, take an inverse Fourier transform:

$$\begin{aligned} T(x, t) &= \mathcal{F}^{-1}(e^{-k^2 t} \hat{f}(k)) \\ &= \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(e^{-k^2 t}) * f(x) \end{aligned}$$

HW: Fourier transform of Gaussian is Gaussian

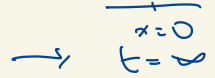
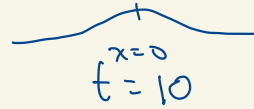
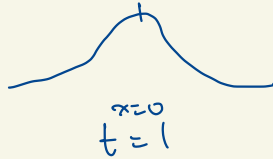
$$= \underbrace{\frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{4t}} e^{-\frac{x^2}{4t}} \right)}_{g(x, t)} * f(x)$$

Whoa! So the temperature distribution at all times
 is determined by initial data $f(x)$ + convolution with $g(x, t)$

$g(x,t) =$  Gaussian, $\int_{-\infty}^{\infty} g(x,t) dx = 1$
 $\text{st. dev} = \sqrt{2t}$

So:

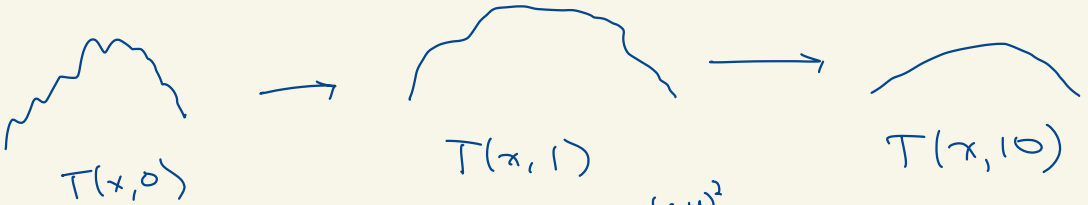
$x=0$
 $t=0$
 $g(x,0) = \delta_0(x)$



- At $t=0$,

$$\begin{aligned}
 T(x,0) &= g(x,0) * f(x) \\
 &= \delta_0(x) * f(x) \\
 &= f(x)
 \end{aligned}$$

but as time goes on, f gets "smeared out":



$$T(x,t) = g * f = \int_{-\infty}^{\infty} \underbrace{\frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4t\pi}}}_{G_t(x,y)} \cdot f(y) dy$$

we call $G_t(x,y)$ a heat kernel. Commonly used in image processing to smooth out pictures

(So, $g * (\cdot)$ is a linear operator! Actually, an integral linear operator. $\int_{-\infty}^{\infty} G_t(x,y) (\cdot) dy$.)

Further Properties of Dirac Deltas (or, the magic of integration by parts)

A more careful defn:

we call such f "test functions".

δ_h is defined by the property

$$\int_{-\infty}^{\infty} \delta_h(x) f(x) dx = f(h)$$

where f is any smooth function where all derivatives vanish at ∞ : $f^{(n)}(x) \rightarrow 0$ as $x \rightarrow \pm\infty$.

This allows us to use integration by parts to meaningfully compute derivatives of δ_h !

Technically, these are called "weak derivatives". δ_h is not a function, but this is a natural generalization of a derivative.

Look: δ'_h is defined by the property that for any test function f ,

$$\int_{-\infty}^{\infty} \delta'_h(x) f(x) dx = \left(\delta_h(x) f(x) \right)_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta_h(x) f'(x) dx$$

integration by parts

$f(\pm\infty) = 0$

$$= -f'(h).$$

But why stop there?

$$\begin{aligned} \int_{-\infty}^{\infty} \delta''_h(x) f(x) dx &= - \int_{-\infty}^{\infty} \delta'_h(x) f'(x) dx \\ &= \left(-\delta'_h(x) f(x) \right)_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \delta'_h(x) f''(x) dx \\ &= f''(h). \end{aligned}$$

$f'(\pm\infty) = 0$

So, this proves $\delta_h^{(n)}$ is the object which has

$$\int_{-a}^a \delta_n^{(n)}(x) f(x) dx = (-1)^n f^{(n)}(x)$$

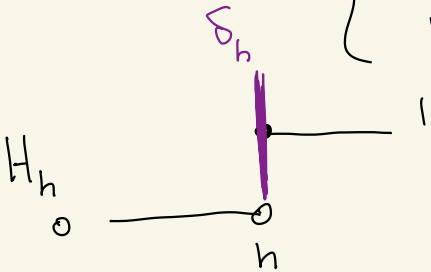
For all test functions f .

$$\chi_{(-\infty, t]} = \begin{cases} 1 & x \leq t \\ 0 & \text{otherwise} \end{cases}$$

• What about integration?

$$\int_{-\infty}^t \delta_h(x) dx = \int_{-\infty}^{\infty} \delta_h(x) \chi_{(-\infty, t]} dx = H_h(t)$$

$$= \begin{cases} 0 & t < h \\ 1 & \text{otherwise} \end{cases}$$



Integral of Dirac Delta

So:

$$H'_h(t) = \delta_h(t)$$

(or, $H'(t-h) = \delta(t-h)$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$)

It is often called the "Heaviside step function".

• What is the Fourier transform of δ_h ?

• From HW 12, Problem 1:

$$\widehat{\delta_h * f} = \widehat{f(x-h)} = e^{-ikh} \widehat{f}$$

But by Fourier transform turns convolution into multiplication,

$$\widehat{\delta_h * f} = \sqrt{2\pi} \widehat{\delta_h} \widehat{f}$$

$\Rightarrow$

$$\widehat{\delta_h}(k) = \frac{e^{-ikh}}{\sqrt{2\pi}}$$

• Incorrect, but useful "proof":

$$\begin{aligned} \widehat{\delta_h}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_h(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-ikh} \end{aligned}$$

From the same problem,

$$\widehat{f}(k-h) = (\tau_h \widehat{f})(k) = e^{ihx} \widehat{f}(k)$$

But

$$\widehat{f}(k-h) = (\delta_h * \widehat{f})(k)$$

Take inverse Fourier transform,

$$\begin{aligned} (e^{ihx} f)(x) &= (\widehat{\delta_h * \widehat{f}})(x) \\ &= (\sqrt{2\pi} \check{\delta_h} f)(x) \end{aligned}$$

$\Rightarrow$

$$\check{\delta_h}(x) = \frac{e^{ihx}}{\sqrt{2\pi}}$$

• Incorrect, but useful "proof"

$$\begin{aligned} \check{\delta_h}(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta_h(k) e^{ikx} dk \\ &= \frac{e^{ihx}}{\sqrt{2\pi}} \end{aligned}$$